Fundamental Solution of 3D Isotropic Elastic Material

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Abstract: In this paper, expression of displacement, shear stress and normal stress of 3D isotropic elastic material are derived. An algorithm of fundamental equation is developed to check the continuity at interface of bonded joint. The Green’s function is used for the fundamental solution of three-dimensional isotropic elastic material. Boundary conditions are applied at the interface of the bonded joint and continuity is checked at interface. The effect of displacements \((u, v, w)\), normal stress \((\sigma_x, \sigma_y, \sigma_z)\) , and shear stress \((\tau_{xy}, \tau_{yz}, \tau_{zx})\) with varying distance is analyzed. Finally, provide the united solution which is suitable for all stable isotropic materials.

Keywords: Fundamental Solution, Isotropic, Bonded Joint, Green’s Function.

1. INTRODUCTION

A material is isotropic if its mechanical and thermal properties are the same in all directions. Isotropic materials can have homogeneous or non-homogeneous microscopic structures. A large number of joints in advanced electronic devices are carried out with the application of these materials. Fundamental solutions are introduced to represent the distribution of displacements and stresses in an elastic medium due to the action of concentrated forces prescribed either at the boundary surface or in the interior of the elastic medium. Fundamental solutions can be used to construct many analytical solutions of practical problems when boundary conditions are imposed [1]. As a useful calculation approach in engineering science, the boundary element method has been greatly developed in recent years. One of the most important points in the theory of the boundary element method is the point force solution or Green's function [2]. Hu [3] obtained the fundamental solutions for an infinite transversely isotropic solid when \(s_1 \neq s_2\), and Pan and Chou [4] also obtained the united solutions for an infinite transversely isotropic solid when \(s_1 \neq s_2\) and \(s_1 = s_2\), but with different forms of constants. Elliott [5] obtained the solution for an infinite medium applied at a point force perpendicular to the plane of isotropy. Kroner [6] obtained the solution for an infinite solid applied at a point force parallel to the plane of isotropy. This paper presents the simplest expression of displacement, shear stress and normal stress of 3D isotropic elastic material and a united solution for isotropic and transversely isotropic material. Therefore, the purpose of this study is to check the continuity of 3D isotropic elastic material joints.

2. MATERIALS AND METHOD

2.1 Basic equations: In the absence of body forces, the governing equation of three dimensional elastic materials is;

\[ \sigma_{ij,j} = 0 \] (1)

Where \(\sigma_{ij}\) is stress tensor. These equations are the elastic equilibrium equation. Constitutive relation for elastic material is expressed by Hooke’s law;

\[ \sigma_{ij} = c_{ijkl}\varepsilon_{kl} \] (2)

Where, \(\varepsilon_{kl}\) is strain tensor, \(c_{ijkl}\) is elastic constant for material. The elastic strain displacement in Cartesian coordinate \(u_i(i = 1, 2)\) are related to the strain by the following relation;

\[ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \] (3)

Where, \(u_{i,j}\) is displacement vector

The characteristics equation of the isotropic elastic material is obtained by solving the given equation.

\[ c_{33}c_{44}s^4 - [c_{11}c_{33} + c_{44} - (c_{13} + c_{44})^2]s^2 + c_{11}c_{44} = 0 \] (4)
Square root of the eigen-equation is denoted by $S_1, S_2$ and $S_3$ is given as below,

$$
S_1 = \sqrt{\frac{(\tilde{c}_{13} - c_{13})(\tilde{c}_{13} + c_{13} + 2c_{44})}{4c_{33}c_{44}}} + \sqrt{\frac{(\tilde{c}_{13} + c_{13})(\tilde{c}_{13} - c_{13} - 2c_{44})}{4c_{33}c_{44}}}
$$

$$
S_2 = \sqrt{\frac{(\tilde{c}_{13} - c_{13})(\tilde{c}_{13} + c_{13} + 2c_{44})}{4c_{33}c_{44}}} - \sqrt{\frac{(\tilde{c}_{13} + c_{13})(\tilde{c}_{13} - c_{13} - 2c_{44})}{4c_{33}c_{44}}}; S_3 = \sqrt{\frac{c_{66}}{c_{44}}}
$$

[Where, $\tilde{c}_{13} = \sqrt{c_{13}c_{33}}$ (5)]

2.2 General Solutions: The general solution for the governing equations of 3D isotropic elasticity has been given by Hu. When force acting on bonded material, the displacement equations $u, v, w$ is given as follows at $s_1 \neq s_2$

$$
u = \sum_{i=1}^{2} \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_i}{\partial y}; \quad v = \sum_{i=1}^{2} \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_i}{\partial x} \quad w = \sum_{i=1}^{2} \frac{\partial \varphi_i}{\partial z_i}; \quad \text{where,} \quad \alpha_1 = \frac{c_{11} - c_{44}s_1^2}{(c_{13} + c_{44})s_1} \quad \text{and} \quad \alpha_2 = \frac{c_{11} - c_{44}s_2^2}{(c_{13} + c_{44})s_2}
$$

(6)

If $\varphi$ is the displacement functions for isotropic material then the normal and shear stress equation at $s_1 \neq s_2$ is given as follows:

$$
\sigma_x = c_{66} \left[ \sum_{i=1}^{2} k_{11} \frac{\partial^2 \varphi_i}{\partial z_i^2} + 2 \left( \sum_{i=1}^{2} \frac{\partial^2 \varphi_i}{\partial x \partial y} \right) \right]
$$

$$
\sigma_y = c_{66} \left[ \sum_{i=1}^{2} k_{11} \frac{\partial^2 \varphi_i}{\partial z_i^2} + 2 \left( \sum_{i=1}^{2} \frac{\partial^2 \varphi_i}{\partial y \partial x} \right) \right]
$$

(7)

$$
\tau_{xy} = 2c_{66} \left( \sum_{i=1}^{2} \frac{\partial^2 \varphi_i}{\partial x \partial y} - \frac{\partial^2 \varphi_i}{\partial y \partial x} \right)
$$

$$
\tau_{xx} = \sum_{i=1}^{2} \omega_i \left( \frac{\partial^2 \varphi_i}{\partial x \partial z_i} + \frac{\partial^2 \varphi_i}{\partial y \partial z_i} \right); \quad \tau_{yy} = \sum_{i=1}^{2} \omega_i \left( \frac{\partial^2 \varphi_i}{\partial y \partial z_i} - \frac{\partial^2 \varphi_i}{\partial x \partial z_i} \right)
$$

(8)

$$
\sigma_z = c_{66} \left[ \sum_{i=1}^{2} k_{21} \frac{\partial^2 \varphi_i}{\partial z_i^2} \right]
$$

(9)

Where, $\xi_i = k_{11}c_{66}; \quad \eta_i = k_{21}c_{66}; \quad k_{31} = (\alpha_1 + s_i)/s_i^2k_{21} = (c_{33} \alpha_i s_i - c_{13})/c_{66}; \quad k_{3i} = (c_{13} \alpha_i s_i - c_{12})/c_{66}$

$$
\omega_1 = k_{31}c_{66}; \quad \omega_3 = c_{44}s_3; \quad \text{[Where,} i = 1, 2]\n$$

2.3 Fundamental Solution: At the interface of the bonded joint consisting of two isotropic material is such that the property of the upper material is equal to that of the property of the lower material. Fundamental solution of isotropic elastic material at the interface of the bonded joint when point force applied at a distance $h$ above the interface in $x$-direction are as follows:

$$
\varphi_i = \frac{D_i x}{\bar{R}_{ii} + s_i |z - h|} + \sum_{j=1}^{2} \frac{D_j x}{R_{ij} + \bar{z}_{ij}}; \quad \varphi_3 = \frac{D_3 y}{R_{33} + s_3 |z - h|} + \frac{D_{33} y}{R_{33} + z_{33}}
$$

(10)

$$
\bar{z}_{ii} = s_i (z - h); \quad \bar{z}_{ij} = s_i + h_j = s_i + s_j h; \quad \bar{z}_{33} = s_3 (z - h)\n$$

(11)

$$
\bar{R}_{ii} = \sqrt{(x^2 + y^2) + s_i^2(z - h)^2}; \quad \bar{R}_{ij} = \sqrt{(x^2 + y^2) + (z_i + h_j)^2}; \quad \bar{R}_{ii}^* = \bar{R}_{ii} + s_i |z - h|; \quad \bar{R}_{ij}^* = (\bar{R}_{ij} + z_{ij})
$$

(12)

$$
R_{33} = \sqrt{r^2 + z_{33}^2}; \quad \bar{R}_{33} = \sqrt{r^2 + \bar{z}_{33}^2}; \quad \bar{R}_{33} = \bar{R}_{33} = \sqrt{(x^2 + y^2) + s_i^2(z - h)^2}\n$$

$$
D_1 = \frac{s_1(c_{11} - c_{44}s_1^2)}{4\pi c_{11}c_{44}(s_1^2 - s_2^2)} T; \quad D_2 = -\frac{s_2(c_{11} - c_{44}s_1^2)}{4\pi c_{11}c_{44}(s_1^2 - s_2^2)} T; \quad D_3 = \frac{T}{4\pi c_{44}s_3}; \quad A_1 = -A_2 = \frac{(c_{13} + c_{44}) T}{4\pi c_{33}c_{44}(s_2^2 - s_1^2)}
$$

(13)

Where, $\varphi_i$ & $\varphi_3$ are displacement functions and $A_i, D_i, R_{ij}, \bar{R}_{ii}, R_{33}, \bar{R}_{33}, \bar{z}_{ij}, s_i, h_i$ are known constants.

$$
u = \sum_{i=1}^{2} \left[ D_i \left( \frac{1}{R_{ii}} \frac{x^2}{R_{ii}^2} \right) + \sum_{j=1}^{2} D_j \left( \frac{1}{R_{ij}} \frac{x^2}{R_{ij}^2} \right) \right] + \left[ \frac{D_3}{R_{33}} \left( \frac{1}{R_{33}} \frac{y^2}{R_{33}^2} \right) + \frac{D_3}{R_{33}^2} \left( \frac{1}{R_{33}} \frac{y^2}{R_{33}^2} \right) \right]
$$

(14)
\[ v = -xy \left( \sum_{i=1}^{2} \left( \frac{D_i}{R_{ii} R_{ii}^*} + \frac{1}{R_{ii} R_{ii}^*} \right) - \frac{D_3}{R_{33} R_{33}^*} + \frac{D_{33}}{R_{33} R_{33}^*} \right) \]  

(15)

\[ w = -x^2 \sum_{i=1}^{2} \alpha_i \left[ \text{sign}(z - h) \frac{D_i}{R_{ii} R_{ii}^*} + \sum_{j=1}^{2} \frac{D_{ij}}{R_{ij} R_{ij}^*} \right] \]  

(16)

\[ \sigma_x = \sum_{i=1}^{2} \xi_i \left[ \left( \frac{D_i}{R_{ii}^2} + \frac{D_{ii}}{R_{ii}^2} \right) - 2C_{66} \left( \sum_{i=1}^{2} \left( \frac{3}{R_{ii} R_{ii}^*} - \frac{2x^2}{R_{ii}^2 R_{ii}^*} - \frac{x^2}{R_{ii}^3 R_{ii}^*} \right) \right) \right] + 2D_{ij} \left( \frac{3}{R_{ij} R_{ij}^*} - \frac{2x^2}{R_{ij}^2 R_{ij}^*} - \frac{x^2}{R_{ij}^3 R_{ij}^*} \right) - D_3 \left( \frac{1}{R_{33} R_{33}^*} - \frac{2y^2}{R_{33}^2 R_{33}^*} - \frac{y^2}{R_{33}^3 R_{33}^*} \right) \]  

(17)

\[ \sigma_y = \sum_{i=1}^{2} \xi_i \left[ \left( \frac{D_i}{R_{ii}^2} + \frac{D_{ii}}{R_{ii}^2} \right) - 2C_{66} \left( \sum_{i=1}^{2} \left( \frac{1}{R_{ii} R_{ii}^*} - \frac{2y^2}{R_{ii}^2 R_{ii}^*} - \frac{y^2}{R_{ii}^3 R_{ii}^*} \right) \right) \right] + 2D_{ij} \left( \frac{1}{R_{ij} R_{ij}^*} - \frac{2y^2}{R_{ij}^2 R_{ij}^*} - \frac{y^2}{R_{ij}^3 R_{ij}^*} \right) - D_3 \left( \frac{1}{R_{33} R_{33}^*} - \frac{2y^2}{R_{33}^2 R_{33}^*} - \frac{y^2}{R_{33}^3 R_{33}^*} \right) \]  

(18)

\[ \tau_{xy} = 2C_{66} \left[ \sum_{i=1}^{2} \left( \frac{1}{R_{ii} R_{ii}^*} - \frac{x^2}{R_{ii}^2 R_{ii}^*} + \frac{x^2}{R_{ii}^3 R_{ii}^*} - \frac{1}{R_{ii} R_{ii}^*} \right) \right] + \left( D_3 \left( \frac{4x^2}{R_{33} R_{33}^*} + \frac{2x^2}{R_{33}^3 R_{33}^*} + \frac{1}{R_{33} R_{33}^*} \right) + D_{33} \left( \frac{4x^2}{R_{33} R_{33}^*} + \frac{2x^2}{R_{33}^3 R_{33}^*} + \frac{1}{R_{33} R_{33}^*} \right) \right) \]  

(19)

\[ \tau_{xz} = -\sum_{i=1}^{2} \omega_i \left[ \text{sign}(z - h) \frac{D_i}{R_{ii} R_{ii}^*} - \frac{x^2}{R_{ii}^2 R_{ii}^*} - \frac{x^2}{R_{ii}^3 R_{ii}^*} \right] + \left( D_3 \left( \frac{1}{R_{33} R_{33}^*} - \frac{y^2}{R_{33} R_{33}^*} - \frac{y^2}{R_{33}^3 R_{33}^*} \right) + D_{33} \left( \frac{1}{R_{33} R_{33}^*} - \frac{y^2}{R_{33} R_{33}^*} - \frac{y^2}{R_{33}^3 R_{33}^*} \right) \right) \]  

(20)

\[ \tau_{yz} = x \sum_{i=1}^{2} \omega_i \left[ \frac{D_i}{R_{ii} R_{ii}^*} + \frac{1}{R_{ii} R_{ii}^*} \right] + \left( D_3 \left( \frac{1}{R_{33} R_{33}^*} + \frac{1}{R_{33} R_{33}^*} \right) + D_{33} \left( \frac{1}{R_{33} R_{33}^*} + \frac{1}{R_{33} R_{33}^*} \right) \right) \]  

(21)

\[ \sigma_z = \sum_{i=1}^{2} \eta_i \left[ \frac{D_i}{R_{ii} R_{ii}^*} + \frac{1}{R_{ii} R_{ii}^*} \right] \]  

(22)

Fundamental solutions of 3D isotropic elastic material at the interface of the bonded joint when point force applied at a distance \( h \) above the interface in \( z \)-direction are as follows:
Similarly, when force applied in upper material, then displacements, \((u', v', w')\), normal stress, \((\sigma_x', \sigma_y', \sigma_z')\), and shear stress, \((\tau_{xy}', \tau_{yz}', \tau_{zx}')\) of lower material has also been derived. Fundamental solutions of 3D isotropic elastic material at the interface of the bonded joint when point force applied at a distance \(h\) below the interface have also been derived.

### 2.4 Material Properties and Boundary conditions

Isotropic elastic material joint comprising of Copper as upper material and Nickel as the lower material is analyzed for this paper. Elastic constants of the material are tabulated below.

<table>
<thead>
<tr>
<th>Material</th>
<th>(c_{11} \times 10^{-7})</th>
<th>(c_{12} \times 10^{-7})</th>
<th>(c_{13} \times 10^{-7})</th>
<th>(c_{33} \times 10^{-7})</th>
<th>(c_{44} \times 10^{-7})</th>
<th>(c_{66} \times 10^{-7})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper (N/m²)</td>
<td>1.684</td>
<td>1.214</td>
<td>1.214</td>
<td>1.684</td>
<td>0.754</td>
<td>0.754</td>
</tr>
<tr>
<td>Nickel (N/m²)</td>
<td>2.508</td>
<td>1.5</td>
<td>1.5</td>
<td>2.508</td>
<td>1.235</td>
<td>1.235</td>
</tr>
</tbody>
</table>

Assume that the two-phase elastic material is perfectly bonded. Therefore the boundary conditions on the interface \((z = 0)\) are given below. Where prime refers to the variables in the lower material, \(z \leq 0\) and the other ones refers to those in the upper material, \(z \geq 0\).
2.5 Model of 3D isotropic elastic material:

\[ u = u' \quad v = v' \quad w = w' \]
\[ \sigma_{xx} = \sigma_{xx}' \quad \tau_{xz} = \tau_{xz}' = \tau_{yz}' \]

Numerical investigation is performed by Green’s function with the application of point force at a distance \( h \) (50mm) above and below the interface in \( x, y \) & \( z \) directions. Dimension of upper and lower material is 100mm\( \times \)100mm\( \times \)100mm when the value of \( z \) varies from -100 mm to 100 mm.

3. RESULTS AND DISCUSSION

Graphical representations of the value of displacement, shear stress and normal stress are given below for both upper and lower material. In all graphs there have a deflection at point 50mm for upper material and -50mm for lower material of the distance, \( z \) where the point force is applied. Displacements, \((u, v, w)\) \& \((u', v', w')\), normal stress, \((\sigma_x, \sigma_y, \sigma_z)\) and shear stress, \((\tau_{xy}, \tau_{xz}, \tau_{yz})\) shows continuity at interface. Normal stress, \((\sigma_x, \sigma_y)\) \& \((\sigma_x', \sigma_y')\) and shear stress, \((\tau_{xy}, \tau_{xz})\) shows discontinuity at interface of 3D isotropic elastic bi-material joint.

![Fig.1 Model of 3D bonded joint](image1)

![Graph2](image2)  
**Fig.2** Distribution of displacement \((u, v \& w)\) against distance \( z \) with force in \( z \) direction of upper material

![Graph3](image3)  
**Fig.3** Distribution of displacement \((u, v \& w)\) against distance \( z \) with force in \( z \) direction of lower material
Fig. 4 Distribution of normal stress ($\sigma_x, \sigma_y, \sigma_z$) against distance $z$ with force in $z$ direction of upper material and also the nature of displacement and stress occurs at the free edge of the material joint than the inner portion of the joint. The possibility to debond and delamination at interface of the isotropic elastic material joints was due to the higher stress and displacement concentration at the free edge.

4. CONCLUSIONS

In this paper, the expression of fundamental equation of 3D isotropic elastic material is derived and algorithm for the fundamental solution is developed by the FORTRAN code. Finally the continuity of the fundamental equation at interface of 3D isotropic elastic material is checked. By using elastic constants and applying point force, the distribution of stress at every point of the bonded material gives the useful information about the permissible amount of force that the material can withstand before its fracture. The higher value of displacement and stress occurs at the free edge of the material joint than the inner portion of the joint. The possibility to debond and delamination at interface of the isotropic elastic material joints was due to the higher stress and displacement concentration at the free edge.

References