AN EXACT SOLUTION OF POST-BUCKLED NONLINEAR BEAM ON AN ELASTIC FOUNDATION

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ABSTRACT
This paper is concerned with buckling problem of flexible beams on an elasticfoundation for free vibration. An exact solution for the post-buckled geometric nonlinear beam with clamed-clamed and clamed-hinged end conditions are presented in this paper. The cubic nonlinearity of the governing equation of motion is induced due to the mid-plane stretching, which is considered in the analysis. The critical buckling load, associated mode shape, the effect of foundation stiffness, andvibration behavior are obtained. The optimum locations of an internal hinge and the optimum buckling force are also investigated for various foundation stiffness of the nonlinear beam on an elastic foundation.

Key words: Buckling; exact solution; hinge; free vibration; nonlinear beam

INTRODUCTION

Buckling is a static instability of structures due to in-plane loading and solving the nonlinear buckling problem for a given axial load results in the post buckling configurations [Nayfeh and Emam 2008]. A plenty of research has been carried out on the subject of buckled beam[Fang and Wickert 1994, Nayfeh et al. 1995, Addessi et al. 2005, Zhang et al. 2005, Li and Batra 2007,Emam and Nayfeh 2009, Xia et al. 2010], plates [Chen 1994, Librescu and Lin 1997] and rods [Whiting 1997, Wang 1997, Li et al. 2002] for many years. Among them, Nayfeh et al [1995] and Chen [1994] were formulated the static buckled configurations to obtain buckled shapes and their associated natural frequencies with the fixed and simply supported post-buckled beams. The governing equations of the nonlinear buckled beams were induced a geometric nonlinearity in the most of the researches. The geometric nonlinearity is due to the mid-plane stretching, which is taken into account in the present study. Fang and Wickert[1994]studied the static deformation of micro-machined beams under prescribed in-plane compressive stress by analytical and experimental means based on geometrically nonlinear imperfect beam. Addessi et al. [2005] investigated free un-damped in-plane vibrations of shear undeformable beams around their highly buckled configuration neglecting rotary inertia effects. Zhang et al.[2005] investigated the secondary bifurcations and tertiary states of a beam resting on nonlinear foundation. They used a three mode Galerkin discretization to produce a set of nonlinear algebraic equilibrium equations and then the algebraic equations were solved by numerically using the root solving and pseudo-arch-length method.
In the recent years, Li and Batra[2007] studied the buckling and the post buckling deformation of uniformly heated pinned-pinned and fixed-fixed Euler-Bernoulli beams supported on linear elastic foundations. They used the shooting method to compute the buckling modes and transitions among them by solving analytically the linear problem. Recently, Xia et al. [2010] investigated the static bending, post buckling and free vibration of nonlinear micro-beams. This study established a nonlinear non-classical Euler-Bernoulli beam model for micro-scale beam by using the modified couple stress theory.
Buckling of column/beam is basic in elastic stability. In some cases beam may have to provide interior joints or internal hinge. The internal hinge may be necessary in designs to facilitate the opening of doors and hatches or other swivel motions. Previously, the buckling force and optimum hinge location on fundamental frequency had beeninvestigated on beams [Wang and Wang 2001, Lee et al. 2003, Cheng et al. 2003] and plates [Xiang et al. 2001, 2003, Gupta and Reddy 2002]. Also,exact vibration solutions of structural members were summarized by Wang et al. [2005]. In the

recent years, buckling of column [Wang 2008] and an infinite beam [Wang 2010] with the internal hinge attached to an elastic foundation have been investigated.Most the works available in the literature for determination of internal hinge location of a beam are the linear vibration problem except Cheng et al. [2003] is a nonlinear random response. Nonlinear vibration of buckling problems of beam with an internal hinge is rare in the literature. The aim of the present study is to determine the optimum location of internal hinge and critical buckling forces at various foundation stiffness of nonlinear beam for clamped-clamped (C – C) and clamped-hinged (C – H).

In this study, an exact solution of the governing differential equation is presented. The geometric nonlinearity is governed due to the mid-plane stretching of the beam; as a result the governing equation is formulated with a cubic nonlinearity. Two types of end conditions of the geometric nonlinear beam such as C – C and C – H on an elastic foundation are taken into account for the analysis of the buckling problem. Exact vibration solutions for internal hinge locations and optimum buckling force corresponding to various foundations stiffness are also investigated for C – C and C – H nonlinearpost-buckled beam on an elastic foundation.

THEORY

The governing equation of motion of nonlinear vibration of the Euler-Bernoulli beam on elastic foundation including the effect of mid-plane stretching [Barari et al. 2011] is as follows

$$EI\hat{W}'''' + m\ddot{\hat{W}} + \hat{\xi}\dot{\hat{W}} + \hat{F}\hat{W}'' + \hat{K}_f\hat{W} - \left(\frac{EA}{2L}\int_0^L (W')^2 d\hat{x}\right)\hat{W}'' = \hat{P}\cos(\hat{\Omega}\hat{t}) \quad (1)$$

Where, the prime indicates the derivative with respect to $\hat{x}$, over dot indicates the derivative with respect to $\hat{t}$ and $\hat{W}$denotes the transverse displacement by the mid-plane stretching of the beam on elastic foundation. The *m* is the mass per unit unreformed length,cross section area A, moment of inertia *I*, length of the beam *L*, Young's modulus of the beam *E*, damping coefficient of the beam ξ, foundation coefficient of modulus K_f, axial force acting on the beam $\hat{F}$, excitation amplitude $\hat{P}$, excitation frequency$\hat{\Omega}$. For the convenience, the following non-dimensional variables are used

$$x = \frac{\hat{x}}{L},\ W = \frac{\hat{W}}{r},\ t = \hat{t}\sqrt{\frac{EI}{mL^4}},\ F = \frac{\hat{F}L^2}{EI},\ \xi = \frac{\hat{\xi}L^2}{\sqrt{mEI}},\ r = \sqrt{\frac{I}{A}},\ K_f = \frac{\hat{K}_f L^4}{EI},\ \text{and}\ P = \frac{\hat{P}L^4}{rEI} \quad (2)$$

Where, *r* is the radius of gyration of the cross section of the beam, therefore, equation (1) can be written as

$$W'''' + \ddot{W} + \xi\dot{W} + FW'' + K_f W - \frac{1}{2}W''\int_0^L (W')^2\, dx = P\sin(\Omega t) \quad (3)$$

The associated boundary conditions for C – C and C - H beam is as follows:

$$W = 0 \text{ and } W' = 0 \text{ at } x = 0, L \quad (4)$$

$$W = 0 \text{ and } W' = 0 \text{ at } x = 0 \quad (5a)$$

$$W = 0 \text{ and } W'' = 0 \text{ at } x = L \quad (5b)$$

BUCKLING FORMULATION

Consider the time dependent, damping factor and force terms are zero, the buckling problem can be obtained from Eq. (3) is as follow

$$W''''(x) + FW''(x) + K_f W(x) - \frac{1}{2}W''(x)\int_0^L (W'(x))^2\, dx = 0 \quad (6)$$

Exact Solution

The integral is constant [Nayfeh and Emam 2008]in Eq. (6) for given *W(x)*. So, consider Qdenotes this constant

$$Q = \frac{1}{2}\int_0^L (W')^2\, dx \quad (7)$$

Substituting Eq.(7) into Eq. (6), the results can be expressed as

$$W'''' + \lambda W'' + K_f W = 0 \qquad (8)$$

where $\lambda = F - Q$ represents the critical buckling load and Eq. (8) is a fourth order ordinary differential equation whose general solution can be expressed into three types [Wang 2008].
Case 1, if $\lambda^2 > 4K_f$, the solution can be written

$$W(x) = C_1 \sin(\alpha x) + C_2 \cos(\alpha x) + C_3 \sin(\beta x) + C_4 \cos(\beta x) \qquad (9a)$$

$$\text{Where}\ \alpha = \sqrt{\frac{\lambda - \sqrt{\lambda^2 - 4K_f}}{2}} \quad \text{and} \quad \beta = \sqrt{\frac{\lambda + \sqrt{\lambda^2 - 4K_f}}{2}} \qquad (9b)$$

Case 2, if $\lambda^2 = 4K_f$, the solution can be written

$$W(x) = C_1 \sin(\alpha x) + C_2 \cos(\alpha x) + xC_3 \sin(\alpha x) + xC_4 \cos(\alpha x) \qquad (9c)$$

$$\text{Where}\ \alpha = \sqrt{\frac{\lambda}{2}} \qquad (9d)$$

Case 3, if $\lambda^2 < 4K_f$, the solution can be written

$$W(x) = C_1 e^{-\alpha x} \sin(\beta x) + C_2 e^{-\alpha x} \cos(\beta x) + C_3 e^{\alpha x} \sin(\beta x) + C_4 e^{\alpha x} \cos(\beta x) \qquad (9e)$$

$$\text{Where}\ \alpha = (K_f)^{\frac{1}{4}} \cos\left(\frac{\theta}{2}\right),\ \beta = (K_f)^{\frac{1}{4}} \sin\left(\frac{\theta}{2}\right),\ \text{and}\ \theta = \pi - \tan^{-1}\left(\frac{\sqrt{\lambda^2 - 4K_f}}{\lambda}\right) \qquad (9f)$$

In this study, we consider only the case 1, $\lambda^2 > 4K_f$, therefore the general solution of the different types of end conditions of the nonlinear beam are as follows:

C – C beam

Applying the boundary condition of Eq. (4) for clamped-clamped beam, we have

$$C_2 + C_4 = 0 \qquad (10)$$

$$\alpha C_1 + \beta C_3 = 0 \qquad (11)$$

$$C_1 \sin(\alpha) + C_2 \cos(\alpha) + C_3 \sin(\beta) + C_4 \cos(\beta) = 0 \qquad (12)$$

$$\alpha C_1 \cos(\alpha) - \alpha C_2 \sin(\alpha) + \beta C_3 \cos(\beta) - \beta C_4 \sin(\beta) = 0 \qquad (13)$$

The determinant of the coefficient matrix of equations (10) – (13) represents the characteristic equation. Therefore, the following characteristic equation can be obtained

$$2\alpha\beta - 2\alpha\beta \cos(\alpha) \cos(\beta) - (\alpha^2 + \beta^2) \sin(\alpha) \sin(\beta) = 0 \qquad (14)$$

The Eigen-values are determined by solving the Eq.(14). Now the mode shapes are given by

$$W(x) = d\left[-\left(\frac{\beta}{\alpha}\right) \sin(\alpha x) - \frac{\alpha \sin(\beta) - \beta \sin(\alpha)}{\alpha(\cos(\alpha) - \cos(\beta))} \cos(\alpha x) + \sin(\beta x) + \frac{\alpha \sin(\beta) - \beta \sin(\alpha)}{\alpha(\cos(\alpha) - \cos(\beta))} \cos(\beta x)\right] \qquad (15)$$

Where, d is a constant to be determined. The expression of mode shapes $W(x)$ governs both the symmetric and anti-symmetric mode shapes. The buckle configuration $W(x)$ satisfies the boundary condition as well as the following condition due the mid stretching of the beam.

$$\lambda = F - Q = F - \frac{1}{2}\int_0^L (W')^2 dx \qquad (16)$$

Substituting Eq. (15) into Eq. (16), a relationship with the variables F, K_f, d and λ are obtained. Therefore, for a given axial load F, the constant d corresponding to any λ can be determined, and subsequently, the mode shapes of beam can be obtained by using the Eq. (15)
Simplify the Eq. (14) the symmetric mode is as follows

$$(\alpha + \beta)\sin\left(\frac{\alpha}{2} - \frac{\beta}{2}\right) + (\alpha - \beta)\sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = 0 \quad (17a)$$

$$\tan\left(\frac{\alpha}{2}\right) = \left(\frac{\beta}{\alpha}\right)\tan\left(\frac{\beta}{2}\right) \quad (17b)$$

Simplify the Eq. (14) the anti-symmetric mode is as follows

$$(-\alpha - \beta)\sin\left(\frac{\alpha}{2} - \frac{\beta}{2}\right) + (\alpha - \beta)\sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = 0 \quad (18a)$$

$$\tan\left(\frac{\alpha}{2}\right) = \left(\frac{\alpha}{\beta}\right)\tan\left(\frac{\beta}{2}\right) \quad (18b)$$

C – H beam

Similarly, satisfying the boundary conditions Eq. (5) the characteristic equation for the clamped-hinged beam can be written as

$$(\alpha^2 - \beta^2)(\alpha\cos(\alpha)\sin(\beta) - \beta\cos(\beta)\sin(\alpha)) = 0 \quad (19)$$

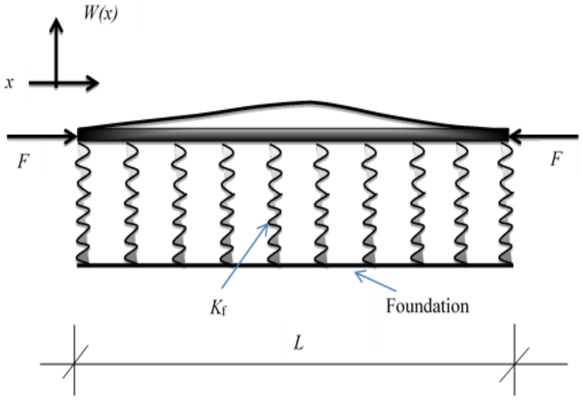
The Eigen-values are determined by solving the Eq. (19). Again the mode shapes yield Eq. (15) and the critical buckling load can be determined after determining the constant *d* by using the Eq. (19).The characteristic equations and Eigen-values for various end conditions of beam on elastic foundation are summarized in Table 1.

Table1 The Eigen values for various end conditions of beam on elastic foundation

End conditions of beam	Eigen values when β=0.3 by using Eqs. (14) and (19) for C – C and C – H beams, respectively
C – C	6.298, 8.971, 12.575, 15.451, 18.858
C – H	4.483, 7.752, 10.889, 14.084, 17.212

RESULTS AND DISCUSSION

The nonlinear post-buckled vibrations of beam on an elastic foundation are analyzed with C – C and C – H. The results are presented in the following Sections with non-dimensional parameters such as length of the beam, axial force, static deflection and foundation stiffness. Figure 1 shows flexible beam on elastic foundation subjected to the axial load, (a) without internal hinge and (b) with internal hinge.



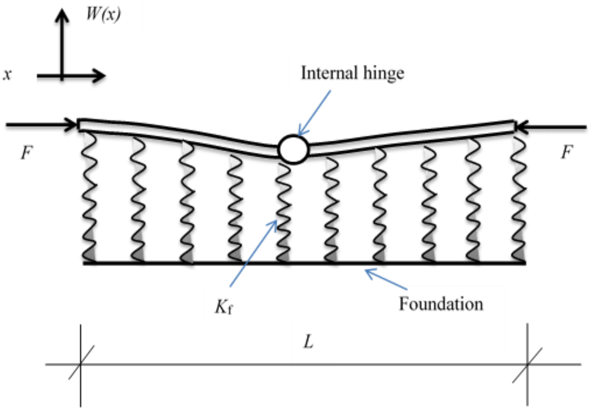
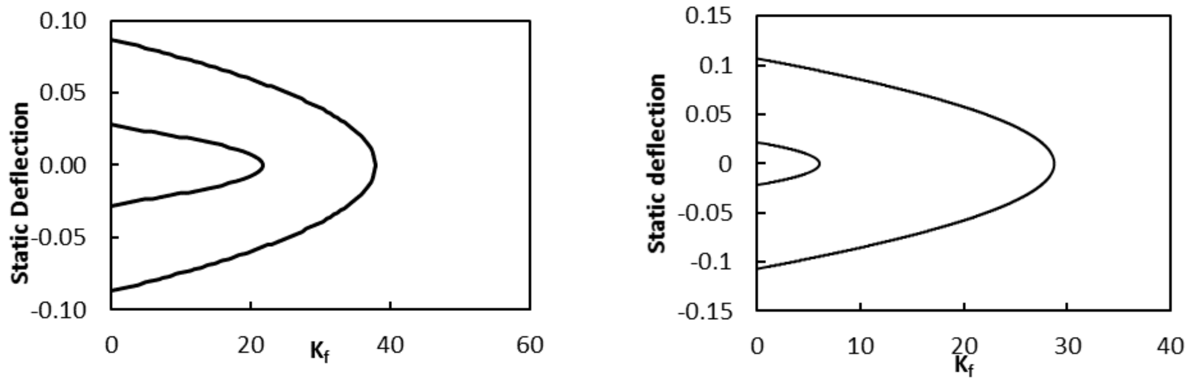


Fig. 1: A flexible beam on elastic foundation subjected to the axial load, (a) without internal hinge and (b) with internal hinge

Fig. 2: illustrates the non-dimensional static deflection with respect to various foundations for C – C and C – Hbeam. The first two modes bifurcation diagrams for non-dimensional static deflection as a functionof non-dimensional foundation stiffness are presented in Figure 2a-b. The non-dimensional static deflections are plotted of the point at x = 0.3L. As the foundation stiffness decreases from the first critical stiffness at K_f = 38.00 and 28.76, the straight position loses stability and therefore, the buckling is formulated for C – C and C – H beam respectively in Figure 2a-b. The stability analysis is not included in this study.



(a) (b)

Fig 2 Non-dimensional static deflection corresponding to foundation stiffness of 1st and 2nd mode (a) C – C beam and (b) C – H beam.

Fig. 3 shows the non-dimensional critical buckling load obtained from exact solution for C – C and C – H beam. The first two modes bifurcation diagrams for non-dimensional deflection as a function of non-dimensional axial force are presented in Figure 3a-b. The non-dimensional static deflections are plotted at x = 0.3L with the foundation stiffness when K_f= 1. As the axial load increase from the 1st mode critical buckling load at F/π^2 = 4.105 and 2.038, the straight position loses stability and thus, the buckling is formulated for C – Cand C – H beam in Figure 3a-b. The stability analysis is not included in this study. It can be seen that the critical axial force are increasing with increasing the foundation stiffness in Table 2a-b for C – C and C – H beam respectively.

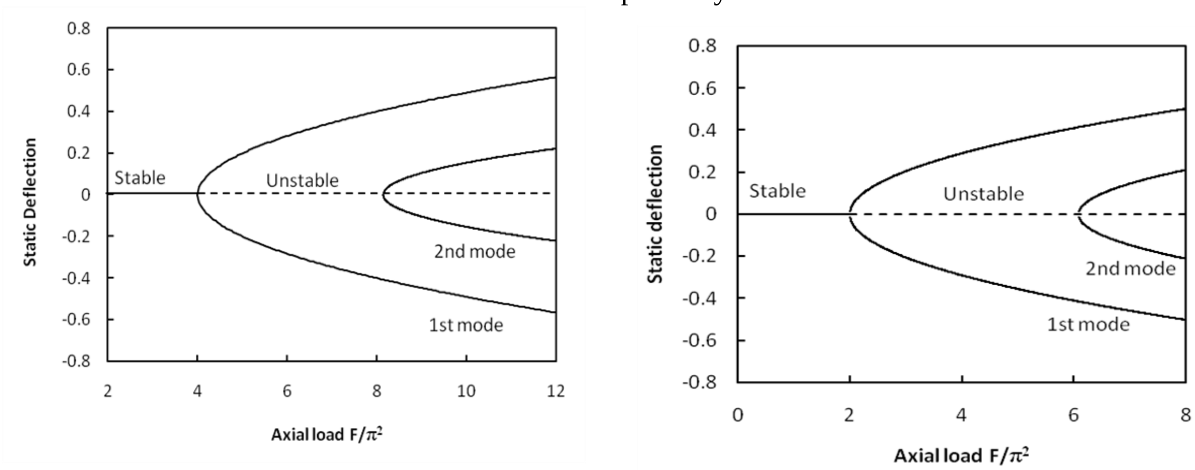


Fig. 3 Non-dimensional static deflection corresponding to axial load of 1st and 2nd mode (a) C – C beam and (b) C – H beam.

Table 2 a The critical buckling force at various K_f for C – C beam.

K_f	Axial load $\frac{F}{\pi^2}$ at $x = 0.3L$		
	1st mode	2nd mode	3rd mode
1	4.015	8.148	16.014
5	4.035	8.158	16.019
10	4.061	8.170	16.026
15	4.086	8.183	16.032
20	4.112	8.195	16.039

Table 2 bThe critical buckling force at various K_f for C – H beam.

K_f	Axial load $\frac{F}{\pi^2}$ at $x = 0.3L$		
	1st mode	2nd mode	3rd mode
1	2.038	6.083	12.006
5	2.078	6.097	12.013
10	2.127	6.113	12.022
15	2.174	6.130	12.030
20	2.221	6.147	12.039

On the other hand, the phase diagram of C – H beam has a non-closed trajectory and the shape of the curve is more likely kidney shape. The nature of the phase diagram of the C – H beam demonstrates that the nonlinearity dominates in the system of C – H beam on elastic foundation at 1st mode vibration. Moreover, the trajectory of the C- H beam originates at the centre of the displacement and velocity; and the trajectory is vertically i.e. velocity axis symmetric. In addition, there is a common or fixed centre of the phase trajectories in the all cases of foundation stiffness for C – H beam.

CONCLUSIONS

An exact solution is presented to solve the nonlinear vibrations of post-buckled beam on an elastic foundation with C – C and C – H end conditions. The effect of foundation stiffness, critical buckling force and interesting vibration behaviors are investigated. The exact vibration solutions for axially loaded nonlinear beams on an elastic foundation with an internal hinge are obtained. The optimum non-dimensional buckling forces are investigated corresponding to the different foundation stiffness for C – C and C – H beam. The result shows that the foundation stiffness is greatly influenced to the buckling force of the beam with an internal hinge. The result obtained from the bifurcation diagrams and the internal hinge locations are useful for practical application with such kind of axially loaded nonlinear beam on an elastic foundation.

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