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STUDY THE EFFECT OF POISSON'S RATIO ON THE INTERFACIAL-STRESS STATE OF BILAYER COMPOSITES USING FINITE DIFFERENCE METHOD

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Abstract- *The bilayer composites are very important fundamental structures of the single walled carbon nanotubes (SWNTs), layered graphene structures, micro-mechanics study of composites, and multi-layered hybrid materials. The interfacial-stress state highly affects the design of such materials. In this paper, a numerical finite difference method (FDM) was used to study the effect of Poisson's ratio on the interfacial-stresses of a bilayer composite material. A displacement potential function was used to formulate the governing equations, displacements, and stresses. Appropriate finite difference schemes were used for the discretization of the bi-harmonic governing equation and the associated boundary conditions. Proper treatment of the boundary conditions was performed at the interface and the corner points of the bilayer. It was found that the Poisson's ratio had a significant effect on the stress-state at the interface as well as inside the materials. Nonlinear behavior and a sudden jump in the stresses were observed in the materials.*

Keywords: FDM, bi-harmonic governing equations, displacement potential function, micro-mechanics, composites

1. INTRODUCTION

Now-a-days composite is a very common word because of its multi-purpose application in many industries such as aerospace, automotive, marine, construction etc [1]–[3]. The word "composite" means 'consisting of two or more distinct parts'. Composites are formed by laying up different materials to form an overall structure that is better than the individual components [4]–[6]. The constituent materials have significantly different physical or chemical properties than the combined layers. The individual layer remains separate and distinct within its domain. In a bilayer composite, there are two materials bonded together having different mechanical properties.

The bilayer composites, such as metal-metal, steel-polymer, concrete-steel etc., having different mechanical properties layer by layer are widely used for modern structures [7]–[9]. They are widely used for single walled carbon nanotubes (SWNTs) and layered graphene structures. In addition, the study of bilayer composite could provide fundamentals of the micro-mechanics analysis of composites, multi-layered hybrid materials [10]–[12]. Therefore, knowing the behavior of bilayer composites under the action of mechanical loading is important for designing the structures. Several methodologies could be used for the solution of the same problem. These methodologies can be classified into three general categories: Experimental, analytical and numerical methods. Usually, the experimental method is used to study the actual behavior. The experimental results are often compared with the numerical simulation results. However, it requires special equipment, testing facilities and thus, often very costly. Analytical solution of a problem is very fast and impressive but sometimes impossible for complex boundary conditions and geometries [13]. The numerical methods have become popular and the ultimate choice of the researchers in the last few decades. Invention and rapid improvement of the computing machine, i.e. sophisticated high-performance computers have accelerated the popularity of the numerical methods.

Stress analysis of bilayer composite requires the solution of partial differential equations. There are various numerical methods available for the solution of partial differential equations. Among them, the most popular methods are Finite Element Method (FEM) and Finite Difference Method (FDM). Finite difference method is an ideal numerical approach for solving partial differential equations. The difference equations that are used to model governing equations in FDM are very simple to code. The global coefficient matrix produced by FDM is a banded structure and is very effective to obtain a good solution. In spite of these characteristics, now-a-days, finite difference method has been used in most of the engineering applications. Finite element method (FEM) is also a popular numerical method for analyzing many elastic problems[14], [15]. The finite element modeling can be efficiently performed with complex boundary shapes. Displacement potential based FDM has been used for many applications [16]–[18]. FEM could produce a reliable result within the body of the structure. However, at certain boundary of the body, FEM may underestimate the critical stress value [19]. In fact, Dow et al. [20] obtained higher accuracy for the stress state at the boundary by using FDM as compared to the FEM. Hossain et al. [21] showed that an efficient approach based on finite difference method for efficient computational effort. It has been shown that the computation time is faster than the other methods. In general, the critical stresses happened in the complex boundaries such as corners, interfaces. The interface-stress is significantly affected by the material properties. The interface-stress is very critical in designing the composite material system [22], [23].

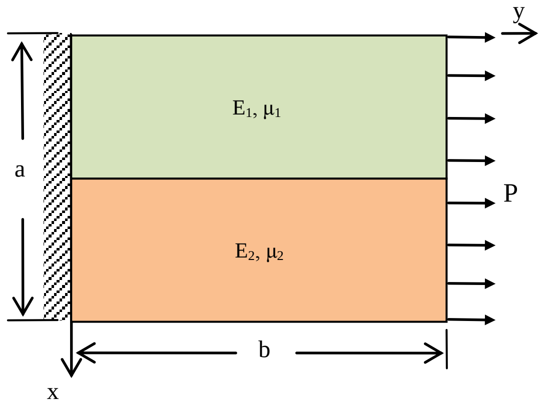
The present work deals with the study of the Poisson's ratio effect on the interface-stress state of a bilayer composite. Finite difference method has been used to discretize the governing equations and the boundary conditions. Appropriate finite difference scheme and special treatment of the interface boundaries has been performed. It was found that the Poisson's ratio has a significant effect on the stress-state at the interface as well as inside the materials. Nonlinear behavior and a sudden jump in the stresses have been observed in the materials.

2. PROBLEM DEFINITION AND THEORY

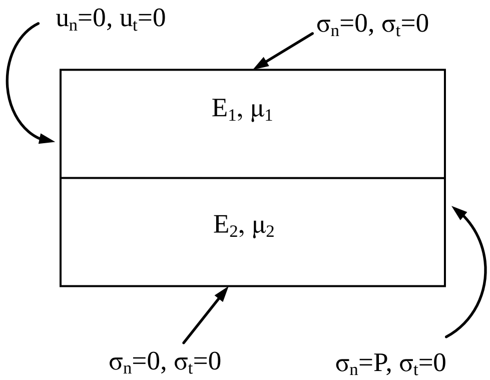
Stress state analysis in a generic material body is usually a three-dimensional (3-D) problem. Fortunately, the most practical problems are often found to conform with the plane stress or plane strain states, hence, the stress analysis of 3-D bodies can easily be treated as a 2-D problem. Let us consider a bilayer composite as shown in **Fig. 1**a with two-dimensional geometry. Each layer is isotropic within its domain but has different material properties from each other. The modulus of elasticity of top and bottom layers are E_1 and E_2, respectively. The Poisson's ratio of top and bottom layers are μ_1 and μ_2, respectively. The numerical results developed in this paper are generic and valid for both plane stress and plane strain cases of the composite. The only difference is - for plane stress, the third dimension of the bilayer can be considered as very thin having no stress components in the thickness direction and for plane strain, the third dimension of the bilayer can be considered as very thick having no strain components in the thickness direction. The bilayer is subjected to a uniform stress on the right side and fixed on the left side, hence, making it a mixed boundary value problem. The boundary conditions of the present problem are illustrated in **Fig. 1**b.

The generic governing equation for 2-D mixed boundary value problem is a bi-harmonic equation [24], i.e.

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=0 \qquad (1)$$



(a)



(b)

Fig. 1: (a) A schematic diagram of a bilayer composite subjected to axial loading, (b) the corresponding boundary conditions

where ψ is the displacement potential function. The equation (1) is valid for both plane stress and plane strain conditions.

The displacements and stresses can be represented in terms of the displacement potential function as follows [25]:

$$u=\frac{\partial^2\psi}{\partial x\partial y} \qquad (2)$$

$$v=-\frac{1}{1+\mu}\left[(1-\mu)\frac{\partial^2\psi}{\partial y^2}+2\frac{\partial^2\psi}{\partial x^2}\right] \qquad (3)$$

$$\sigma_x=\frac{E}{(1+\mu)^2}\left[\frac{\partial^3\psi}{\partial x^2\partial y}-\mu\frac{\partial^3\psi}{\partial y^3}\right] \qquad (4)$$

$$\sigma_y=-\frac{E}{(1+\mu)^2}\left[\frac{\partial^3\psi}{\partial y^3}+(2+\mu)\frac{\partial^3\psi}{\partial x^2\partial y}\right] \qquad (5)$$

$$\sigma_{xy}=\frac{E}{(1+\mu)^2}\left[\mu\frac{\partial^3\psi}{\partial x\partial y^2}-\frac{\partial^3\psi}{\partial x^3}\right] \qquad (6)$$

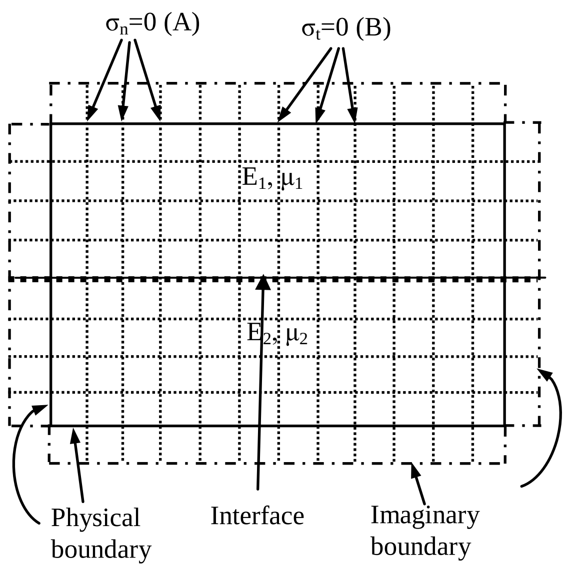
The Eq. (2)-(6) are valid for both plane stress or plane strain condition. The solution of the governing equation (Eq. (1)) would give the displacement potential function at every point in the material and then Eq. (2)-(6) can be used for displacement and stress distributions in the material. However, no analytical solution exists for Eq. (1) under the prescribed boundary conditions in **Fig. 1**b. Hence, we seek for a numerical method for an approximate solution to the problem. In this paper, we used a finite difference method to solve the problem.

3. NUMERICAL MODELING

To apply the finite difference method the domain of the problem has been discretized into suitable mesh size. The discretization of the bilayer is illustrated in **Fig. 2**. A convergence study has been performed to obtain the optimum mesh size of the discretization. An additional imaginary boundary is needed around the physical boundary to properly apply the governing equations at all inner nodes of the material. The boundary conditions have been applied at the boundary. At each boundary node two conditions apply, for example, top traction-free boundary has σ_n=0, σ_t=0. The both conditions are satisfied at the boundary nodes but one is assigned to the physical nodes and another is applied for the imaginary nodes. Using the Taylor series expansion, the governing equation can be discretized and expressed as Eq. (7)

$$zk1\{\psi(i-2,j)+\psi(i+2,j)\} - zk2\{\psi(i-1,j)+\psi(i+1,j)\} - zk3\{\psi(i,j+1)+\psi(i,j-1)\} + zk4.\psi(i,j) + zk5\{\psi(i-1,j-1)+\psi(i-1,j+1)+\psi(i+1,j-1)+\psi(i+1,j+1)+\psi(i,j-2)+\psi(i,j+2)\} = 0 \quad (7)$$

where, $zk1 = r^4$, $zk2 = 4(r^4+r^2)$, $zk3 = 4(1+r^2)$, $zk4 = (6r^4+8r^2+6)$, $zk5 = 2r^2$.



A- satisfied at the inner physical boundary but assigned for imaginary nodes

B- satisfied at the inner physical boundary but assigned for physical nodes

Fig. 2: Representative FDM discretization scheme: there are physical nodes and imaginary nodes outside the physical boundary. The boundary conditions are satisfied at the physical boundary but can be assigned to physical and imaginary nodes.

The stencil representation of the governing equation is illustrated in **Fig. 3**a. In a similar manner, the displacement and traction boundary conditions can be expressed in discretized form. The discretized equations are derived in such a way that the stencil of the each boundary condition is confined within the layer. A representative form of the displacement boundary equation at the interface of the bilayer is illustrated in **Fig. 3**b. Note that the nodes associated with the interface pivot point for the top layer are staying within the top layer. Same happened for the bottom layer. In this way, all the mixed boundary conditions have been specially treated at the interface as well as in the corner points.
The detail of the numerical treatment of the boundary conditions is discussed in ref. [26]

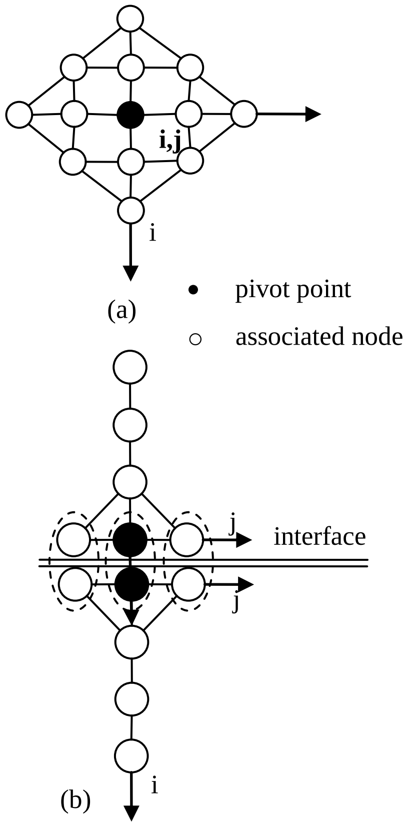


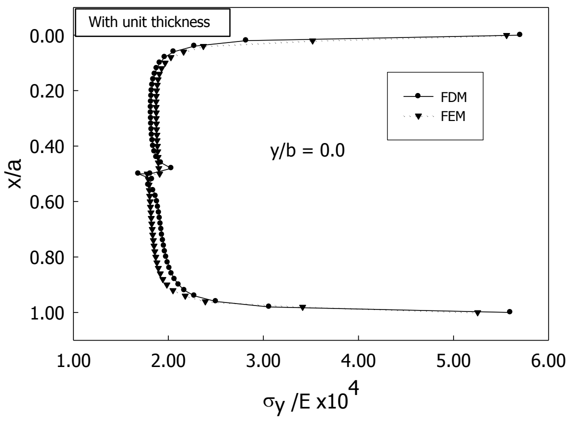
Fig. 3: Representative of the (a) stencil for the governing equations for the inner nodes of the material (b) stencil for the displacement component, u_y at the interface of the bilayer composite.

4. RESULTS AND DISCUSSION

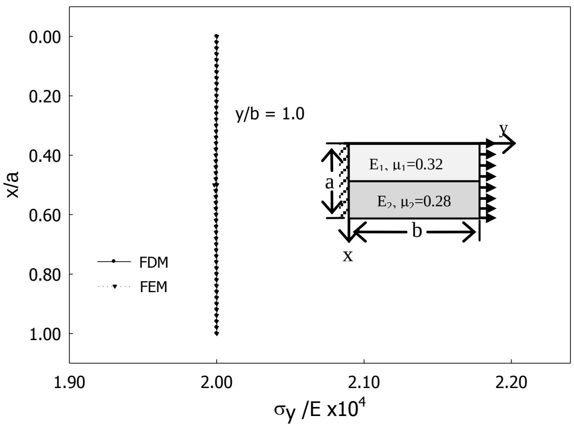
The analytical expressions of the differential equations were formulated by using a computer numerical code (FORTRAN language). A system of linear algebraic equations was formed in terms of displacement potential functions ψ as an unknown at each nodal points. Then the system of equations was solved by using Gaussian elimination method. At first, the finite difference results were verified with FEM results. Then the FDM results to study the effect of the Poisson's ration on the interface stress distributions are discussed.

4.1 Model Verification with FEM

A similar bilayer composite model has been setup by using a FEM software package for a particular set of material properties in the top and bottom layer. The following boundary conditions were used for both FDM and FEM model for comparison. At the left side of the bilayer, $u_n=0$, $u_t=0$; at the right side $\overline{\sigma}_n = \sigma^{yo} = \sigma_1/E_1 = \sigma_2/E_2 = 1.5x10^{-4}$, $\overline{\sigma}_t = 0.0$; at the top and bottom surfaces $\overline{\sigma}_n = 0.0$, $\overline{\sigma}_t = 0.0$, where, σ^{yo} is the dimensionless stress; σ_1, σ_2 is the applied stress, E_1, E_2 are the modulus of elasticity, $\mu_1=0.32$, $\mu_2=0.28$ are the Poissons' ratios of the upper and lower material, respectively.



(a)



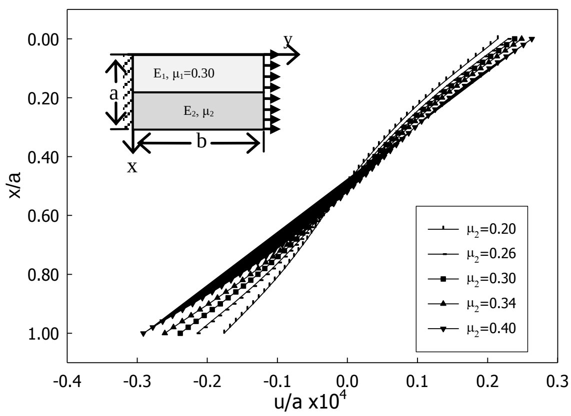
(b)

Fig. 4: Comparison between FEM vs FDM approach: normalized normal stress (σ_y/E) distribution at (a) y/b =0.0, (b) y/b =1. Please note: for top layer, $E=E_1$, and for bottom layer, $E=E_2$.

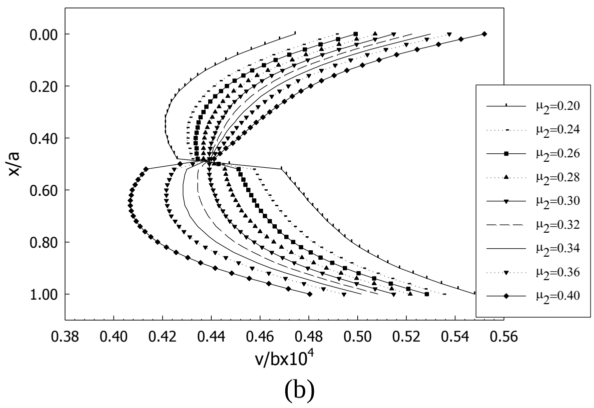
A square geometry of the bilayer was considered for both cases having a/b=1.0 and a=b=25 unit. This problem is solved for stress and displacement distribution by using finite difference method and finite element method taking. The FDM results were found to be in good agreement with FEM results. The normalized normal stress (σ_y/E) distribution comparison is shown in **Fig. 4**a,b as a representative illustration. The stress components were chosen at (a) y/b =0.0, (b) y/b =1. Please note: for top layer, $E=E_1$, and for bottom layer, $E=E_2$. It was found that the FDM results were very good agreement with FEM results.

4.2 Effect of the Poisson's Ratios

A bilayer composite has a fixed boundary at the left side and is subjected to a uniform normalized normal stress $\overline{\sigma}_y = \sigma_1/E_1 = \sigma_2/E_2 = \sigma^{yo} = 1.5x10^{-4}$ on the right side (in **Fig. 1**, $P = \overline{\sigma}_y$). Note that the stress component is normalized by the modulus of elasticity of respective layer. This form is very advantageous since we can apply the final results for any modulus of elasticity of interest. Also, note that the displacement and stress relation in Eq. (2)-(6) can easily be normalized by the modulus of elasticity. The top and bottom surfaces of the bilayer are traction-free. An FDM model was set up under the above mentioned boundary conditions. Then the model was solved for different combinations of the Poisson' ratio of the two layers. Considering the practical applicability, the Poisson's ratio were varied for 0.2 to 0.4 since most of the materials have the Poisson's ratio in this range. For simplicity of demonstrating the results, the Poisson's ratio variation in the bottom layer has been discussed here. The Poisson's ratio of the top layer was kept constant to $\mu_1 = 0.3$ which corresponds to the most common engineering material. In this paper, the results are presented in non-dimensional form. Hence, the results are generic and applicable to any geometry.



(a)



(b)

Fig. 5: (a) Variation of normalized displacement component (u/a) and (b) Variation of normalized displacement component (v/b) for different Poisson's ratios of lower material at y/b=0.24 of the bilayer composite.

The variation of normalized displacement components, u/a and v/b for different Poisson's ratios is illustrated in **Fig. 5**a,b. The geometric configuration of the bilayer is shown in an inset figure for better interpretation of the results. These results were extracted from a particular segment (y/b=0.24) of the bilayer composite. It was found that the variation of the Poisson's ratio in one layer affected the displacement distribution at another layer. The larger Poisson's ratio caused larger displacement component u in both layers. But for displacement component v had a different trend in top and bottom layer. For displacement component v, the larger Poisson's ratio caused larger value in top layer but smaller value in bottom layer.

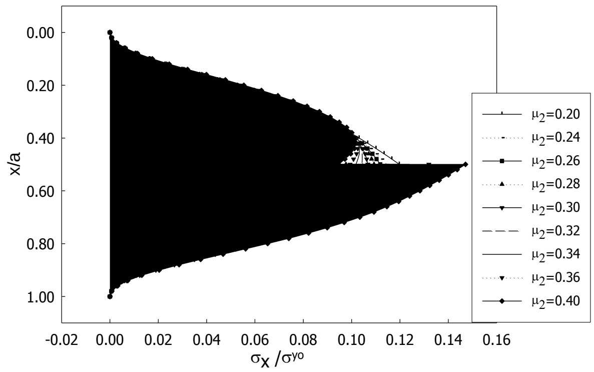


Fig. 6: Variation of normalized normal stress (σ_x / σ^{yo}) for different Poisson's ratios of lower material at y/b=0.24 of the bilayer composite

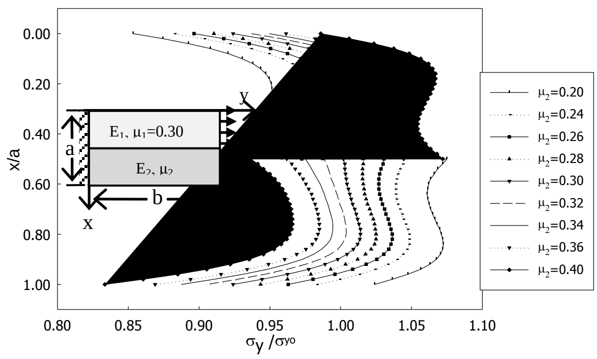


Fig. 7: Variation of normalized normal stress (σ_y / σ^{yo}) for different Poisson's ratios of lower material at y/b=0.24 of the bilayer composite

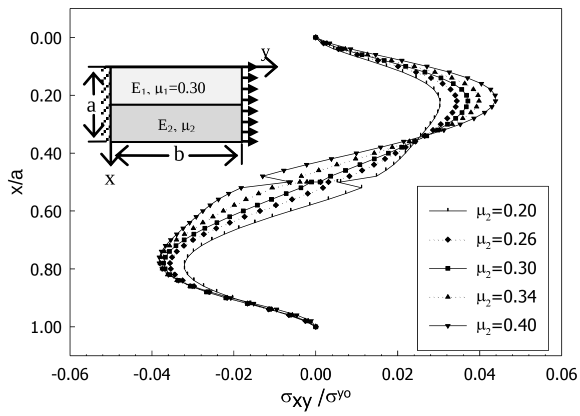


Fig. 8: Variation of normalized shear stress ($\sigma_{xy} / \sigma^{yo}$) for different Poisson's ratios of lower material at y/b=0.24 of the bilayer composite.

The normalized stress distributions are illustrated in **Fig. 6**, **Fig. 7**, and **Fig. 8**. The normal stress, σ_y and shear stress, σ_{xy} distributions are affected in both top and bottom layer although the Poisson's ratio of bottom layer was changed. But the normal stress, σ_x in the top layer was not affected much by the variation of the Poisson's ratio of bottom layer. Also note that the shear stress, σ_{xy} at the top and bottom faces were found to be zero which verifies the applied traction-free boundary condition at the top and bottom.

For all combinations of the Poisson's ratios, the interface-stress state was affected. A jump in the normal stresses can be observed from the stress distribution plots in **Fig. 6** and **Fig. 7**. But the normalized shear stress was found to be continuous over the interface of the bilayer composite. However, when the different modulus of elasticities would be inserted in the normalized shear stress results there would be a jump in the actual shear stress.

In case of same materials at the top and bottom, the stress-jump would be diminished and become continuous stress distributions as expected.

5. CONCLUSION

The finite difference method (FDM) approach can accurately predict the displacement and stress state of a bilayer composite. The interfacial-stress state is highly affected by the different Poisson's ratios of the layers. Appropriate use of the difference equation is the key to the FDM solution for the composite layer. Proper treatment of the boundary conditions was performed at the interface and the corner points of the bilayer. It was found that the variation of the Poisson's ratio in one layer affected the displacement and stress distribution at another layer. The larger Poisson's ratio caused larger displacement component u in both layers. But for displacement component v had a different trend in top and bottom layer. Nonlinear behavior and a sudden jump in the stresses were observed at the interface of the bilayer composite material.

6. FUTURE WORK

This approach can be extended to the multilayered composite materials. Various boundary conditions and geometry of the model can be considered for future investigation.

7. ACKNOWLEDGEMENT

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9. NOMENCLATURE

Symbol	Meaning	Unit
σ	Stress	(Pa)
u, v	Displacement	(m)
ψ	Displacement potential function	(m^3)
E	Youngs' modulus of elasticity	(Pa)
μ	Poisson's ratio	(dimensionless)