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NUMERICAL MODELING OF A MEMORY- BASED RADIAL DIFFUSIVITY EQUATION USING CAPUTO DEFINITION OF FRACTIONAL ORDER DERIVATIVE

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Abstract- *Inclusion of fractional order derivative, that comes into diffusion equation to incorporate the effect of history of rock, fluid, and fluid flow, makes the equations complicated and difficult to solve both analytically and numerically compared to conventional diffusivity equation. In this paper, a numerical model for a time-fractional non-linear diffusion equation is developed utilizing Caputo definition for fractional order derivative. To validate the numerical model, the diffusion equation is solved analytically using Caputo definition of fractional order derivative for Dirichlet boundary condition and for an initial condition. Numerical and analytical solutions are compared and it is found that numerical and analytical solutions match with negligible error.*

Keywords: Memory, Numerical Modeling, Fractional diffusion equation, Caputo definition, Uniform mesh.

1. INTRODUCTION

The three most important technological challenges in the current world- sustainable use and management of the earth's groundwater, reduction of concentration of greenhouse gases in atmosphere and petroleum production are associated with fluid flow through porous media. Modeling and simulation of porous media flow is crucial to overcome the challenges. Numerous models and way to look at the subsurface flow phenomena has been developed over more than the past fifty years. Recent years have seen interest in investigation the effect of history of rock, fluid, and flow on flow through porous media. In this perspective, all materials are assumed to have memory, and their past is considered to affect their present and future characteristics. Memory incorporation make the governing equations intricate and solving the equations become more challenging.

The 'memory' idea is relatively new and growing in petroleum engineering. Hossain and Abu-Khamsin [1] defined memory as the effect of past events on the present and future course of developments. Hossain *et al.* [2] think that the memory of fluid is the most important and most neglected feature in considering fluid flow models. In this direction, Hossain *et al.* [3] proposed the following diffusivity equation:

$$\frac{\partial}{\partial x}\left[\frac{\rho k}{\mu} t^{\alpha} \frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\frac{\partial p}{\partial x}\right)\right] = \rho \phi c_t \frac{\partial p}{\partial t} \quad (1)$$

Fractional order derivative comes in the mathematical model when memory is incorporated. Derivation of numerical solutions to fractional order differential equation is challenging because of its non-local behaviour. However, number of studies on the numerical approaches to fractional diffusion equation have been increasing in the literature. Shen *et al.* [4] proposed an explicit finite difference approximation for space fractional diffusion equation. Tadjeran *et al.* [5] used Crank–Nicholson method combining with spatial extrapolation to obtain temporally and spatially second-order accurate numerical estimates for fractional diffusion equation. Sun *et al.* [6] solved time-fractional diffusion equations applying a semi-analytical finite element method. Sweilam *et al.* [7] applied Crank-Nicolson finite difference method to solve time-fractional diffusion equation. Variational iteration method and the Adomian decomposition method were used by Monami *et al.* [8] to solve linear fractional partial differential equations. Celik and Duman [9] applied Crank-Nicolson method with the Riesz fractional derivative to numerically solve fractional diffusion equation.

Instead of treating fractional order derivative by their definition and discretizing the term that contains fractional order derivative Hossain *et al.* [3] considered the term as a parameter and then solved the equation numerically in the way the integer order partial differential equation is solved. Hence their numerical solution is not correct in the exact mathematical sense. The solution of a fractional order diffusivity equation is

important because it shows the way to solve other fractional order diffusivity equations.

In this paper, Hossain *et al.*'s [3] model is solved numerically for Caputo definition of fractional order derivative. Implicit finite difference approximation for the computation of Caputo's fractional derivative given by Murio [10] is used to discretize the Eq. (1).

The mathematical model is discretized using uniform mesh both in space and time. For some positive integer X, M, T and N the grid sizes in space and time are defined by $h = X/M$ and $\Delta t = \frac{T}{N}$ respectively. The grid points in the space interval $[0, X]$ are the numbers $x_i = ih, i = 0,1,2, \dots, M$, and the grid points in the time interval $[0, T]$ are labeled $t_n = n\Delta t, n = 0,1,2, \dots, N$. The values of the function p at the grid points are denoted $p_i^n = p(x_i, t_n)$ and $p_i = p(x_i)$ respectively.

2. NUMERICAL SOLUTION

Substitution of $\frac{\rho k}{\mu} t^\alpha$ by C_1, $\rho\phi c_t$ by C_2 and use of Caputo definition for fractional order derivative gives,

$$\frac{\partial}{\partial x}\left[C_1(x_i, t_n)\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial p}{\partial x}\right)\right]_i^n = C_2(x_i, t_n)\frac{\partial p_i^n}{\partial t} \quad (2)$$

Substitution of $\left[C_1\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial p}{\partial x}\right)\right]_i^n$ by F in Eq. (2) and discretization of the equation gives,

$$\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) = C_2(x_i, t_n)\frac{p_i^n - p_i^{n-1}}{\Delta t} \quad (3)$$

Murio's (2008) approximation formula for Caputo definition of fractional order derivative,

$$\frac{\partial^\alpha u(x_i,t_n)}{\partial t^\alpha} = \sigma_{\alpha,\Delta t}\sum_{j=1}^n \omega_j^{(\alpha)}(u_i^{n-j+1} - u_i^{n-j}) \quad (4)$$

where, $\sigma_{\alpha,\Delta t} = \frac{1}{\Delta t^\alpha \Gamma(2-\alpha)}$ and $\omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}$. Applying Murio's formula $F_{i+\frac{1}{2}}^n$ and $F_{i-\frac{1}{2}}^n$ can be written as,

$$F_{i+\frac{1}{2}}^n = \frac{1}{\Delta x}C_1\left(x_{i+\frac{1}{2}}, t_n\right)\sigma_{\alpha,\Delta t}\left[p_{i+1}^n - p_i^n - p_{i+1}^{n-1} + p_i^{n-1} + \sum_{j=2}^n \omega_j^{(\alpha)}\left(p_{i+1}^{n-j+1} - p_i^{n-j+1} - p_{i+1}^{n-j} + p_i^{n-j}\right)\right] \quad (5)$$

$$F_{i-\frac{1}{2}}^n = \frac{1}{\Delta x}C_1\left(x_{i-\frac{1}{2}}, t_n\right)\sigma_{\alpha,\Delta t}\left[p_i^n - p_{i-1}^n - p_i^{n-1} + p_{i-1}^{n-1} + \sum_{j=2}^n \omega_j^{(\alpha)}\left(p_i^{n-j+1} - p_{i-1}^{n-j+1} - p_i^{n-j} + p_{i-1}^{n-j}\right)\right] \quad (6)$$

Substitution of Eq. (5) and (6) in Eq. (3) and rearrangement gives,

$$-C_1\left(x_{i-\frac{1}{2}}, t_n\right)p_{i-1}^n + \left[C_1\left(x_{i+\frac{1}{2}}, t_n\right) + C_1\left(x_{i-\frac{1}{2}}, t_n\right) + \frac{C_2(x_i,t_n)\Delta x^2}{\sigma_{\alpha,\Delta t}\Delta t}\right]p_i^n - C_1\left(x_{i+\frac{1}{2}}, t_n\right)p_{i+1}^n = \frac{C_2(x_i,t_n)\Delta x^2}{\sigma_{\alpha,\Delta t}\Delta t}p_i^{n-1} + C_1\left(x_{i+\frac{1}{2}}, t_n\right)G_i^n - C_1\left(x_{i-\frac{1}{2}}, t_n\right)H_i^n \quad (7)$$

where

$$G_i^n = -p_{i+1}^{n-1} + p_i^{n-1} + \sum_{j=2}^n \omega_j^{(\alpha)}(p_{i+1}^{n-j+1} - p_i^{n-j+1} - p_{i+1}^{n-j} + p_i^{n-j}) \quad (8)$$

$$H_i^n = -p_i^{n-1} + p_{i-1}^{n-1} + \sum_{j=2}^n \omega_j^{(\alpha)}(p_i^{n-j+1} - p_{i-1}^{n-j+1} - p_i^{n-j} + p_{i-1}^{n-j}) \quad (9)$$

Eq. (7) is written for each grid respectively and then the system of equation is solved. Here a problem arises to solve the equation. The pressures, the solution of the equation, depends on the calculation of density, permeability, viscosity, porosity, and compressibility which themselves depend on these pressures. To get rid of this dilemma an iterative scheme to update density, permeability, viscosity, porosity, and compressibility is used. The approach is illustrated qualitatively by:

(Density, permeability, viscosity, porosity, compressibility data)$^{n,\,z-1}$ (Pressures)$^{n,\,z}$ = RHS$^{n,\,z-1}$

At any given iteration level $(z-1)$ in each time step n, the pressure, density permeability, viscosity, porosity, and compressibility data are assumed known from the most recent computational value. The most recent value at the start of a new time step is the old-time step value and during a given time step is the last iteration. The coefficients are updated using the new values of pressure as the pressures are solved and this process is continued. The iteration process terminates when the convergence criterion is satisfied. MATLAB code is written based on Eq. (7) to numerically solve the Eq. (1).

3. ANALYTICAL SOLUTION

$C_1 = C_2 = 1$ is considered to find the analytical solution of Eq. (1). For $C_1 = C_2 = 1$ the equation becomes linear. Initial and boundary condition are taken as $p(x,0) = \sin(\pi x)$, and $p(0,t) = p(1,t) = 0$ respectively.

Utilizing Caputo definition for fractional order derivative the analytical solution of Eq. (1) is found as:

$$p(x,t) = \sin(\pi x) \quad 0 < \alpha < 1 \quad (10)$$

For $\alpha = 0$ in Caputo case, analytical solution of Eq. (1) is not derived. Instead, analytical solution of the equation $(\partial/\partial x)[(\partial^\alpha/\partial t^\alpha)(\partial p(x,t)/\partial x)] - \pi^2\sin(\pi x) = (\partial p/\partial t)$ with the same initial and boundary condition is obtained as:

$$p(x,t) = \sin(\pi x)\, e^{-\pi^2 t} \quad (11)$$

4. RESULTS AND ANALYSIS

4.1 Validation of Numerical Models

Numerical solutions are compared with analytical solutions to validate the numerical models. Initial condition $p(x,0) = \sin(\pi x)$, and boundary condition $p(0,t) = 0, p(1,t) = 0$ are used for numerical solution as they are used in analytical solution.

Analytical solution (Eq. 10) found using Caputo

definition is applicable for $0 < \alpha < 1$ and the numerical solution for this definition is applicable for $0 \leq \alpha < 1$. The term '$-\pi^2 \sin(\pi x)$' is added in right hand side of Eq. 7 as forcing function while comparing with the analytical solution (Eq. 11) of the modified equation in the case of $\alpha = 0$. MATLAB code is written for numerical solutions and analytical solutions. The solutions are shown and compared in Fig. 1, Fig. 2, Fig. 3, and Fig. 4 for $\alpha = 0$, $\alpha = 0.25$, $\alpha = 0.50$, and $\alpha = 0.75$ respectively. Numerical solutions match very well with analytical solutions. For $\alpha \neq 0$, the solution does not depend on time, hence only one curve is found for analytical and numerical solutions.

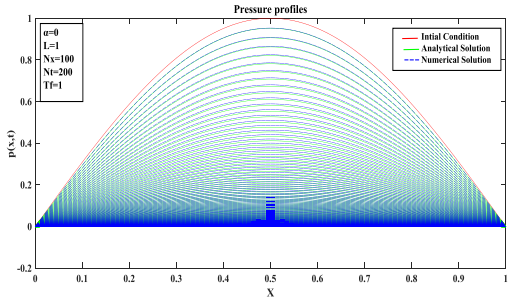

Fig.1: Comparison between analytical and numerical solution for $\boldsymbol{\alpha = 0}$.

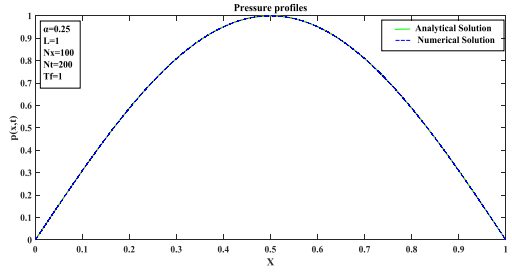

Fig.2: Comparison between analytical and numerical solution for $\boldsymbol{\alpha = 0.25}$.

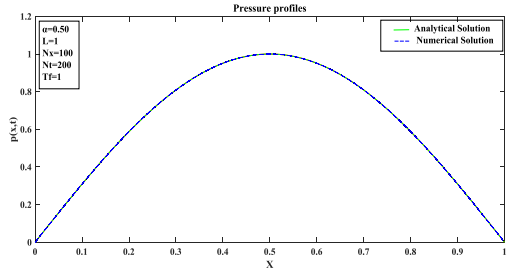

Fig.3: Comparison between analytical and numerical solution for $\boldsymbol{\alpha = 0.50}$.

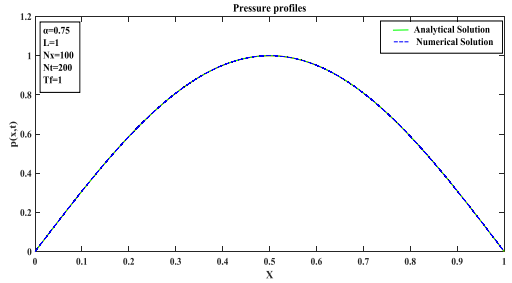

Fig.4: Comparison between analytical and numerical solution for $\boldsymbol{\alpha = 0.75}$.

5. CONCLUSIONS

A Numerical model is developed to solve a time-fractional diffusion equation applying Caputo definition for fractional order derivative. Finite difference method has been applied. The fractional diffusion equation is discretized using uniform mesh both in space and time. Analytical solution is derived for Dirichlet boundary condition and for an initial condition. The numerical model is validated comparing with the analytical solutions.

6. ACKOWLEDGEMENT

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8. NOMENCLATURE

Symbol	Meaning	Unit
c_t	Total compressibility of the system	(atm^{-1})
k	Permeability of the medium	(darcy)
L	Length in the x-direction	(cm)
M	Number of grid points in space	Dimensionless
N	Number of grid points in time	Dimensionless
Nx	Number of grid cells	Dimensionless
Nt	Number of time steps	Dimensionless
P	Pressure	(atm)
T	Time at which pressure is calculated	(sec)
Δt	Length of time step	(sec)
Tf	Final time	(sec)
α	Fractional order of differentiation	Dimensionless
μ	Fluid dynamic viscosity	(cp)
ρ	Fluid density	(g/cm^3)
ϕ	Porosity of the fluid media	fraction